Towards Material Modelling in Physical Models Using Digital Waveguides

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Abstract

Di ital Wave uides have been used extensively for musical instrument and room acoustics modellin . They can be used to form simplistic models for ideal wave propa ation in one, two and three dimensions. Models in 1-d for strin

we find that waves travel with a speed which varies with frequency accordin to $c(w) = \sqrt{w/a}$, where $a = \sqrt{\frac{\rho A}{EI}}$. Thus wave speed increases (from zero) with frequency. By addin a bar like term to (3) we may represent a stiff strin as follows.

$$\frac{\partial^2 y}{\partial t^2} = \frac{F}{\rho}\frac{\partial^2 y}{\partial x^2} - \frac{EI}{\rho A}\frac{\partial^4 y}{\partial x^4}, \qquad (5)$$

where now the restorin forces are due to tension and bendin stiffness. A ain by considerin harmonic solutions to (5) we may derive a expression for the frequency dependent wave speed in a stiff strin .

$$c(w)^2 = \frac{F}{2\rho} + \sqrt{\left(\frac{F}{2\rho}\right)^2 + \frac{\rho A}{EI}w^2}. \qquad (6)$$

Note that when the stiffness is removed that (6) reduces to the case of the ideal strin , and that when all tension is removed we reduce to the case of the ideal beam. We also note that for low frequencies the speed approximates that of the strin , but that as frequency increases the speed of wave travel becomes more bar like. Finally, we observe that the introduction of a constant tension to a beam results in a non-zero wave speed at zero frequency.

2.2 A tring on a Viscoelastic Foundation

In this short section we show how it is possible to add new terms to the PDE's described previously in order to introduce dispersion and frequency dependent loss. Firstly we consider placin a strin on a purely elastic foundation, which may be thou ht of as layin the strin on bed of sprin s (Graff, 1975). The overnin equation is now

$$F\frac{\partial^2 y}{\partial x^2} - Gy = \rho\frac{\partial^2 y}{\partial t^2}, \qquad (7)$$

where the new parameter is the foundation stiffness G. By a ain tryin harmonic solutions we find the followin relationship between frequency and wave number,

$$w^2 = c_0^2\left(k^2 + \frac{G}{F}\right), \qquad (8)$$

where $c_0 = \sqrt{F/\rho}$ is the wave speed in the absence of foundation stiffness. By considerin frequency a ainst wave number we are able to predict both the fundamental frequency, and then the relative positions of each subsequent harmonic. A raph of this relationship is shown in Fi ure 1 and shows that the fundamental increases with frequency and that as the wavenumber increases, the resonant peaks will approach a harmonic series equivalent to the strin in the absence of foundation stiffness.

We now propose a damped strin obtained by includin a resistive force to the motion resultin in the followin overnin equation,

$$F\frac{\partial^2 y}{\partial x^2} - g\frac{\partial y}{\partial t} = \rho\frac{\partial^2 y}{\partial t^2}, \qquad (9)$$

where g is the resistive coefficient (Graff, 1975). This process can be thou ht of in the same terms as for the elastic

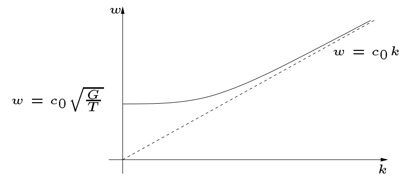


Figure 1: Frequency against wavenumber for a string on an elastic sub-base.

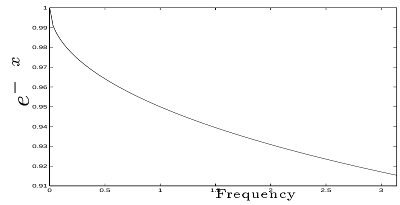


Figure 2: Freq-dependent Damping for String on Viscous Sub-Grade.

foundation, only with dash-pots replacin sprin s. This time however the dampin prohibits the free propa ation of harmonic waves, however we may consider solutions of the form $y = Ae^{-\ x}e^{j(kx-wt)} = Ae^{j[(k+j\ )x-wt]}$. Thus we have dispersive travellin waves which also include frequency dependent dampin . olvin (9) for these solutions yields

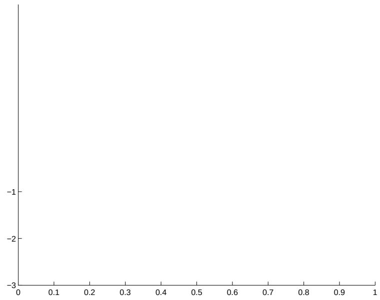
$$k = M^{1/2}\cos(\phi/2),\ \alpha = M^{1/2}\sin(\phi/2), \qquad (10)$$

for

$$M = \frac{w}{F}\left(g^2 + \rho^2 w^2\right)^{1/2},\ \phi = \tan^{-1}\left(\frac{g}{\rho w}\right).$$

We see that w will dominate for lar e values of the frequency, so that in any practical situation, the dispersion is minimal at low frequencies, and ne li ible elsewhere. We should also note that the dampin term will cause dampin of a lowpass nature. hown in Fi ure 2 is the dampin term for some typical simulation values.

3 Incorporating Bending Stiffness into Waveguide Models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1

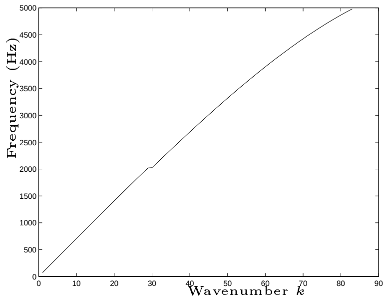

Figure 8: Frequency versus wavenumber for DWN bar model.

attached with impedance R_s. Be innin with the junction velocity equation for junction j, we have

$$
\begin{aligned}
V_j(n) &= \frac{2}{R}\left[V^+_{j,1}(n) + V^+_{j,2}(n) + R_s V^+_{j,s}(n)\right] \\
&= \frac{2}{R}\left[V^-_{j-1,2}(n-1) + V^-_{j+1,1}(n-1) - R_s V^-_{j,s}(n-1)\right] \\
&= \frac{2}{R}\left[V_{j-1}(n-1) + V_{j+1}(n-1) - R_s V_j(n-1)\right] \\
&- \frac{2}{R}\left[V^+_{j-1,2}(n-1) + V^+_{j+1,1}(n-1) - R_s V^+_{j,s}(n-1)\right] \\
&= \frac{2}{R}\left[V_{j-1}(n-1) + V_{j+1}(n-1) - R_s V_j(n-1)\right] \\
&- \frac{2}{R}\left[V^-_{j,1}(n-2) + V^-_{j,2}(n-2) + R_s V^-_{j,s}(n-2)\right] \\
&= \frac{2}{R}\left[V_{j-1}(n-1) + V_{j+1}(n-1) - R_s V_j(n-1)\right] \\
&- V_j(n-2),
\end{aligned}
$$

which is equivalent to

$$
V_j(n+1) - 2V_j(n) + V_j(n-1) = \frac{2}{R}\left[V_{j-1}(n) - 2V_j(n) + V_{j+1}(n)\right] - 4\frac{R_s}{R}V_j(n),
$$

where $R = 2 + R_s$ is the total junction impedance. Now, recallin equation (7) we may write a FD for the system as

$$
V_j(n+1) - 2V_j(n) + V_j(n-1) = \mu\frac{F}{\rho}\left[V_{j-1}(n) - 2V_j(n) + V_{j+1}(n)\right] - T^2\frac{G}{\rho}V_j(n),
$$

where T is the time step, Δ is the spatial step, with $\mu = \frac{T^2}{\Delta^2}$. Thus we must fix

$$
\begin{aligned}
\frac{2}{R} &= \frac{\mu F}{\rho}, \\
\frac{4R_s}{R} &= \frac{GT^2}{\rho}. \qquad (17)
\end{aligned}
$$

By fixin the time step T, then solC1I9T19msolC1W1H1BPC1IMtrueIDEIBTvH1BPC1reT191Tf30 .1-9901.1Tf30 .1-9901.1(s)TjETq1

1 and this time settin the third impedance to R_d we may follow a similar formulation as before. This time we find that

$$V_j(n+1) - 2V_j(n) + V_j(n-1) = \frac{2}{R}\left[V_{j+1}(n) - 2V_j(n) + V_{j-1}(n)\right] - \frac{2R_d}{R}\left[V_j(n) - V_j(n-1)\right]$$